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Problem Session to the Course: Mathematics I

Environmental and Resource Management WS 2002/03

Solutions to Sheet No. 13 (Deadline: January, 27/28 2002)

Homework

H 13.1: Eigenvalues of A: $\lambda_1 = -1, \lambda_{2/3} = 1$

Eigenvectors:

$\lambda_1 = -1$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \vec{x} = 0, \quad \vec{x} = (0,1,-1)^T, \quad \vec{c}_1 = \frac{1}{\sqrt{2}}(0,1,-1)^T$$

$\lambda_{2/3} = 1$:

$$\vec{c}_2 = \frac{1}{\sqrt{2}}(0,1,1)^T, \quad \vec{c}_3 = (1,0,0)^T$$

$$C = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

$$q(\vec{x}) = \vec{x}^T A \vec{x} = x_1^2 + 2x_2x_3.$$

$$B = C, \quad C^T A C = \mathrm{diag}(1,-1,-1)$$

$$q(C\vec{y}) = (C\vec{y})^T A (C\vec{y}) = \vec{y}^T C^T A C \vec{y} = \vec{y}^T \mathrm{diag}(1,-1,-1)\vec{y} = y_1^2 - y_2^2 - y_3^2.$$

H 13.2: $$\det(A - \lambda E) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 5 \\ 0 & -1 & -1-\lambda \end{pmatrix} =$$

$$(1-\lambda)[(1-\lambda)(-1-\lambda)+5] = (1-\lambda)[4+\lambda^2] = 0$$

$$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_{2/3} = \pm 2i$$

Eigenvectors:

$$\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 5 \\ 0 & -1 & -1-\lambda \end{pmatrix} \vec{x} = \vec{0}$$

$\lambda_1 = 1$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & -1 & -2 \end{pmatrix}, \quad \vec{x} = t(1,0,0)^T$$

$\lambda_1 = 2i$:

$$\begin{pmatrix} 1-2i & 1 & 1 \\ 0 & 1-2i & 5 \\ 0 & -1 & -1-2i \end{pmatrix} \vec{x} = \vec{0}$$

$$\begin{pmatrix} 1-2i & 1 & 1 \\ 0 & -1 & -1-2i \\ 0 & 0 & 0 \end{pmatrix} \vec{x} = \vec{0}, \quad \vec{x} = \left(\frac{2i}{1-2i}, -1-2i, 1 \right)^T$$

$\lambda_1 = -2i$:

$$\begin{pmatrix} 1+2i & 1 & 1 \\ 0 & 1+2i & 5 \\ 0 & -1 & -1+2i \end{pmatrix} \vec{x} = \vec{0}$$

$$\begin{pmatrix} 1+2i & 1 & 1 \\ 0 & -1 & -1+2i \\ 0 & 0 & 0 \end{pmatrix} \vec{x} = \vec{0}, \quad \vec{x} = \left(\frac{-2i}{1+2i}, -1+2i, 1 \right)^T$$

H 13.3: $\det(A - \lambda E) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 1 = \lambda^2 - 4\lambda + 2.$

Eigenvalues:

$$\lambda_{1/2} = 2 \pm \sqrt{2}$$

Eigenvectors: $\lambda = 2 + \sqrt{2}$

$$\begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix} \vec{x} = \vec{0}, \quad \vec{x} = \frac{1}{\sqrt{4-2\sqrt{2}}} (-1+\sqrt{2}, 1)^T$$

$\lambda = 2 - \sqrt{2}$

$$\begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{pmatrix} \vec{x} = \vec{0}, \quad \vec{x} = \frac{1}{\sqrt{4+2\sqrt{2}}} (-1-\sqrt{2}, 1)^T$$

$$B\begin{pmatrix} \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}$$

$$B^{-1}=B^T,\quad B^{-1}AB=\begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{pmatrix}$$

Additional Problems

P 13.1: $$\det(A-\lambda E)=\det\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}=$$

$$(3-\lambda)\left((5-\lambda)^2-4=\lambda^2-10\lambda+21\right)=0$$

$$\Rightarrow \quad \lambda_1=7,\quad \lambda_{2,3}=3.$$

$$\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}\vec{x}=\vec{0},\qquad \vec{x}_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\vec{x}=\vec{0},\qquad \vec{x}_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\qquad \vec{x}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$C^TAC=\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix} 14 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}=\begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

b)

$$q(\vec{x})=\vec{x}^TA\vec{x}=5x_1^2+4x_1x_2+5x_2^2+x_3^2$$

$$\overline{q}(\vec{y}):=q(C\vec{y})=q(\vec{x})=\vec{y}^T\begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}\vec{y}=7y_1^2+3y_2^2+3y_3^2$$

The transformation $\vec{x}=C\vec{y}$ is an orthogonal transformation of the coordinate system. $\det C=1 \Rightarrow$ the transformation is a rotation of the coordinate system.

$$C=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}=$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) & 0 \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection about $x_2 = x_1$ and a rotation by $\frac{\pi}{4}$ around the origin.

P 13.2: $C = (\vec{c}_1, \ldots, \vec{c}_n), \quad \vec{c}_i^T \vec{c}_j = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}, \quad A\vec{c}_i = \lambda_i \vec{c}_i.$

$$C^T AC = C^T (A\vec{c}_1, \ldots, A\vec{c}_n) = C^T (\lambda_1 \vec{c}_1, \ldots, \lambda_n \vec{c}_n) =$$

$$\begin{pmatrix} \lambda_1 \vec{c}_1^T \vec{c}_1 & \ldots & \lambda_n \vec{c}_1^T \vec{c}_n \\ & \ldots & \\ \lambda_1 \vec{c}_n^T \vec{c}_1 & \ldots & \lambda_n \vec{c}_n^T \vec{c}_n \end{pmatrix} = diag(\lambda_1, \ldots, \lambda_n).$$

P 13.3: The quadratic form q is given by

$$q(\vec{x}) = x_1^2 - 2x_1x_2 + 2x_2^2 = \vec{x}^T A\vec{x} = \vec{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \vec{x}$$

Eigenvalues of A

$$(1 - \lambda)(2 - \lambda) - 1 = 1 - 3\lambda + \lambda^2 = 0$$

$$\lambda_{1/2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

Eigenvectors:

$$\lambda = \frac{3}{2} + \frac{\sqrt{5}}{2} \quad \Rightarrow \vec{c}_1 = \frac{1}{\sqrt{\frac{10}{4} - \frac{\sqrt{5}}{2}}} \left(\frac{1}{2} - \frac{\sqrt{5}}{2}, 1 \right)^T$$

$$\lambda = \frac{3}{2} - \frac{\sqrt{5}}{2} \quad \Rightarrow \vec{c}_1 = \frac{1}{\sqrt{\frac{10}{4} + \frac{\sqrt{5}}{2}}} \left(\frac{1}{2} + \frac{\sqrt{5}}{2}, 1 \right)^T$$

$$C = (\vec{c}_1, \vec{c}_2)$$

$$C^T AC = \text{diag} \left(\frac{3}{2} + \frac{\sqrt{5}}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2} \right)$$

$$q(C\vec{y}) = (C\vec{y})^T AC\vec{y} = \vec{y}^T C^T AC\vec{y} = \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right) y_1^2 + \left(\frac{3}{2} - \frac{\sqrt{5}}{2} \right) y_2^2$$

Problems can be downloaded from the internet site:
http://www.math.tu-cottbus.de/~pawell/education/erm/erm.html